ON MEAN APPROXIMATION OF HOLOMORPHIC FUNCTIONS BY RATIONAL FUNCTIONS

by David Bell

In this paper, which is an extension of Bers [3], we will consider results of which the following is a special case: Let α and β be two smooth, simple closed curves in the complex plane C such that α is inside β (α may meet β), and let V be the (open) set of points inside β and outside α. (By a smooth curve we will mean one which is twice continuously differentiable as a function of its arc length.)

Theorem. If $S \subset \alpha$ then the following two statements are equivalent:
i) $M(S)$, the linear space of all rational functions whose poles are simple and are contained in S, is dense in the Banach space of holomorphic L_p functions on V for $1 \leqq p < 2$.

ii) $$\int_\alpha \log \rho(z,\beta) ds = -\infty,$$

or

$$\int_\alpha \log \rho(z,S) ds = -\infty,$$

where

$$\rho(z,H) = \inf\{|z-w| : w \in H\},$$

and where ds denotes integration with respect to arc length.

More generally, unless otherwise stated, we will from now on understand V to denote some fixed bounded open subset of the complex plane such that every component X of C-clos V is bounded by a smooth, simple closed curve. A boundary point of V which is not in the boundary of some such X will be called a cut point of V. Let us assume that the set of all cut points of V can be expressed as the union of a finite number of smooth curves. As a descriptive convenience we assume that C-clos V has finitely many components, and that V is connected. Such a V is depicted in Figure 1.

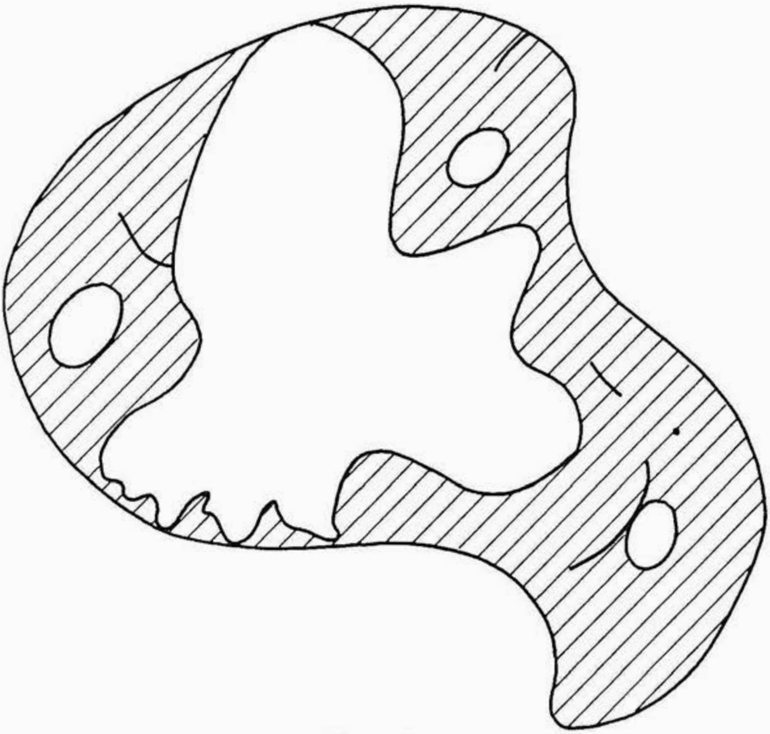


Figure 1.

Let $A_p(V)$ be the Banach space of L_p holomorphic functions on V, where $1 \leqq p < 2$, under the usual L_p norm

$$\| f \|_{V,p} = \left[\iint_V |f(z)|^p dx\, dy \right]^{1/p}.$$

Where the discussion is equally applicable to any such p and V, we may just write A and $\| \ \|$ for the preceding.

Certainly, if $M(S)$ is dense in A, then S must be dense in the cut points of V (at least for the domains which we are considering, whose boundary has planar measure zero). And since the closure of $M(S)$ in A is the same as the closure of $M(\operatorname{clos} S)$ in A, let us assume, unless otherwise stated, that S is a closed subset of $C - V$ which contains all cut points of V. (Here $\operatorname{clos} H$ denotes the closure of H, and $\operatorname{int} H$ denotes the interior of H, for $H \subset C$.)

Call two components X and Y of C-$\operatorname{clos} V$ equivalent if there exists a sequence of components of C-$\operatorname{clos} V$,

$$W_1 = X,\ W_2, \cdots, W_n = Y$$

such that

$$\int_{\partial W_j} \log \rho(z, W_{j+1}) ds = -\infty$$

for $j = 1, 2, \cdots, n-1$. (It will be shown later that this is an equivalence relation.)

The principal results are:

Proposition 1. $M(S)$ is dense in A if given a component X of C-clos V, there exists an equivalent component Y which satisfies at least one of the following conditions:

i) Y is unbounded,

ii) $\int_{\partial Y} \log \rho(z, S) ds = -\infty$,

iii) $\sum \rho(z, \partial Y) = \infty \qquad (z \in S \cap Y)$.

Proposition 2. If X is a bounded component of C-clos V, then

$$M(S) \cap \{f : \|f\| < 1\}$$

is a normal family on

$$(V \cup \operatorname{clos} X) - S$$

whenever all of the following conditions hold:

i) $S \cap \operatorname{clos} X \subset \partial X$,

ii) $\int_{\partial X} \log \rho(z, S) ds > -\infty$,

iii) X is not equivalent to any other component of C-clos V.

The following converse statements to Proposition 1 may be deduced from Proposition 2:

Corollary 1. If $M(S)$ is dense in A, then given a component X of C-clos V, there exists an equivalent component Y of C-clos V such that either Y is unbounded, or clos Y contains infinitely many points of S, or some point of clos$Y$ is an accumulation point of the set of cut points of V.

Corollary 2. In order that the polynomials be dense in A, it is necessary and sufficient that V have no cut points and that all components of C-clos V be equivalent.

Corollary 3. If $S \subset \partial V$, and if no two different components of C-clos$V$ are equivalent, then $M(S)$ is dense in A if and only if

$$\int_{\partial X} \log \rho(z, S) ds = -\infty \tag{1}$$

for all bounded components X of C-clos V.

If there exist two or more equivalent components of C-clos V, then it is possible for $M(S)$ to be dense in A even if (1) does not hold. For example, consider a domain V bounded by

$$\{z: |z| = 3\} \cup \{z: |z-1| = 1\} \cup \alpha,$$

where α is a smooth, simple closed curve,

$$\alpha \cap \{z: |z-1| = 1\} = \{0\},$$

and

$$\int_{-\pi}^{\pi} \log \rho(1 + e^{it}, \alpha)\, dt = -\infty,$$

as depicted in Figure 2. Let $z_n = 1 + \exp(it_n)$, where $t_1, t_2, t_3, \cdots$ is a monotone, strictly increasing sequence of positive numbers whose limit is π. We will see that α may be chosen so that $M(\{z_1, z_2, z_3, \cdots\})$ will be dense in $A_p(V)$.

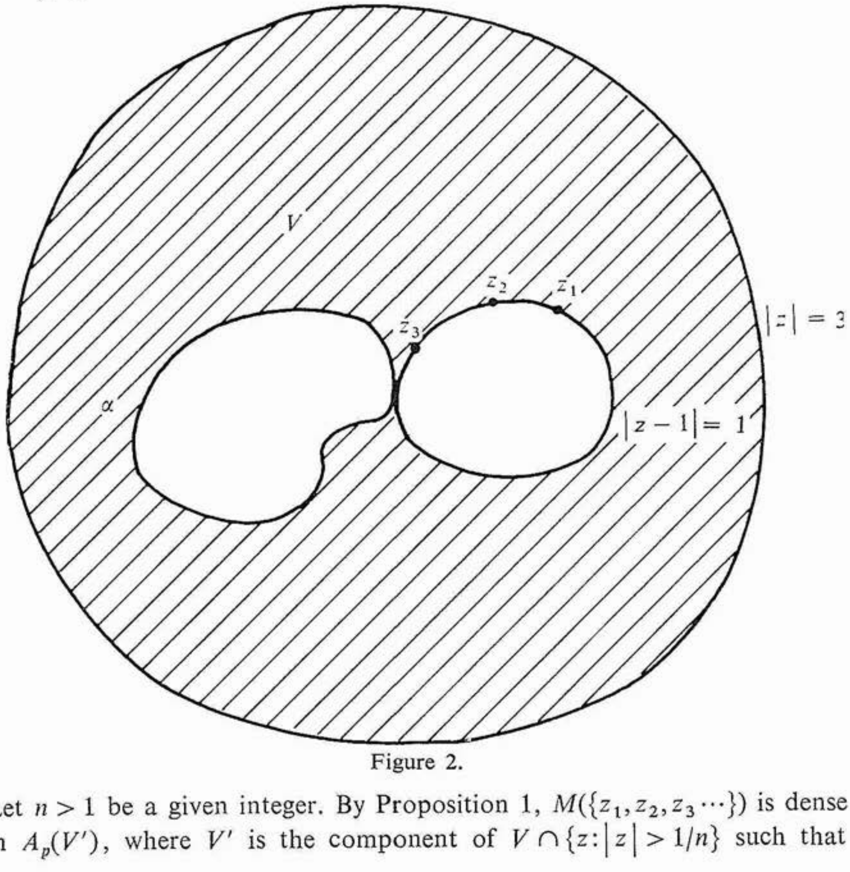


Figure 2.

Let $n > 1$ be a given integer. By Proposition 1, $M(\{z_1, z_2, z_3 \cdots\})$ is dense in $A_p(V')$, where V' is the component of $V \cap \{z: |z| > 1/n\}$ such that

$3 \in \partial V'$. Then there exists a smooth modification β of α in $\{z: |z| < 1/n\}$ which is contained in $\operatorname{clos} V$, which meets $\{z: |z| < 1/n\}$ at and only at $z = 0$, which agrees with α in $\{z: |z| \geqq 1/n\}$, and for which there exist

$$f_1, f_2, \cdots, f_n \in M(\{z_1, z_2, z_3, \cdots\})$$

such that

$$\| f_j - 1/(z + 1/j - 1) \|_{W,p} < 1/n \qquad (j = 1, 2, \cdots, n),$$

where W is the domain bounded by $|z| = 3$, $|z-1| = 1$, and β. By an obvious induction in which we successively modify α only in smaller and smaller neighborhoods of $z = 0$, we obtain a limiting curve γ and a corresponding domain D bounded by $|z| = 3$, $|z-1| = 1$, and γ, such that $1/(z + 1/j - 1)$ is in the closure of $M(\{z_1, z_2, z_3 \cdots\})$ in $A_p(D)$ for $j = 1, 2, 3, \cdots$, and γ is a smooth, simple closed curve which meets $|z-1| = 1$ at and only at $z = 0$, and

$$\int_{-\pi}^{\pi} \log \rho(1 + e^{it}, \gamma) dt = -\infty .$$

Then by Proposition 1,

$$M(\{1 - 1/j : j = 1, 2, 3, \cdots\})$$

is dense in $A_p(D)$, so $M(\{z_1, z_2, z_3, \cdots\})$ must also be dense in $A_p(D)$.

Proposition 3. Let X be a bounded component of C-$\operatorname{clos} V$ which is bounded by a smooth, simple closed curve α, where X is not equivalent to any other component of C-$\operatorname{clos} V$. Suppose that

$$\int_\alpha \log \rho(z, S) ds > -\infty .$$

It follows from the proof of Proposition 2 that there exists a smooth, simple closed curve δ in $\operatorname{clos} X$ which passes through all points of $S \cap \alpha$, and satisfies

$$\int_\alpha \log \rho(z, \delta) ds > -\infty .$$

Suppose further, that δ can be chosen so that every point of $S \cap \operatorname{clos} X$ lies either inside or on δ. Then

$$M(S) \cap \{f : \| f \| < 1\}$$

is a normal family on

$$V \cup (\operatorname{clos} X) - S .$$

Corollary 4. If C-clos V has a component X for which S and X satisfy all the hypotheses of Proposition 3, then $M(S)$ is not dense in A.

The Cardinality of S. Assume in this section that V has no cut points.

1. Let S_0 be a (finite) set such that for each equivalence class $W_1, W_2, \cdots, W_n$ of bounded components of C-clos V, S_0 has exactly one point which meets $\cup$ clos W_j $(j = 1,2,\cdots,n)$. Then by Proposition 1 there exists a countable subset S of ∂V such that all accumulation points of S are in S_0, and $M(S)$ is dense in A.

2. A subset S of ∂V may have uncountably many points in clos X for each component X of C-clos V, but nevertheless, $M(S)$ may fail to be dense in A. For example, let $V = \{z : 1 < |z| < 2\}$, and let u_n be a monotone, strictly increasing sequence of positive numbers whose limit is π, and such that

$$\int_{-\pi}^{\pi} \log \rho(e^{it}, S)\, dt > -\infty$$

$$(S = \{\exp(iu_n) : \quad n = 1,2,3,\cdots\}).$$

One can rearrange the disjoint open intervals which are the components of $\{z : |z| = 1 \text{ and } z \notin S\}$ so that the complement S' in $\{z : |z| = 1\}$ of the rearranged intervals is a perfect (hence uncountable) set. Clearly

$$\int_{-\pi}^{\pi} \log \rho(e^{it}, S) dt = \int_{-\pi}^{\pi} \log \rho(e^{it}, S') dt$$

so by Corollary 3, $M(S)$ cannot be dense in A.

A similar construction is valid in arbitrary domains.

3. Let $X_1, X_2, \cdots, X_n$ be a sequence of components of C-clos V, where exactly one has been chosen from each equivalence class, except that none has been included from the collection of all components equivalent to the unbounded one. Consider a countable subset S of $X_1 \cup X_2 \cup \cdots \cup X_n$ such that S has infinitely many points in each X_j. By Proposition 1, S may be chosen so that $M(S)$ is dense in A, or by Corollary 4, S may be chosen so that $M(S)$ is not dense in A. In particular, if S has a limit point in every X_j (hence also, if S has uncountably many points in each X_j), then $M(S)$ is dense in A.

Extensions. Versions of all the preceding results are easily seen to hold if V is not connected, or if weaker conditions are placed on ∂V, or if our norms are weighted by suitable powers of the Bergman kernel function (see Bell [*1*]). Proofs are similar.

Lemma 1. *Let f and g be real valued, continuously differentiable functions defined for $|t| < \varepsilon$ for some $\varepsilon > 0$, and suppose that for such t, $f(t) \leqq g(t)$, $f(0) = g(0)$, and $f'(0) = 0 = g'(0)$. Let $\alpha(t) = t + if(t)$ and let $\beta(t) = t + ig(t)$ for $|t| < \eta$. Then for sufficiently small $\eta > 0$,*

$$\int_\alpha \log \rho(z, \beta) ds > -\infty$$

if and only if

$$\int_{-\eta}^{\eta} \log |f(t) - g(t)| dt > -\infty .$$

Proof of Lemma 1. If t is sufficiently close to 0, then there exists a point $u(t)$ such that $|u(t)| < \varepsilon$ and

$$\rho(\alpha(t), \beta) = |\alpha(t) - \beta(u(t))| .$$

Differentiating $|\alpha(t) \cdot \beta(t)|^2$ in u at $u(t)$, we see that

$$(u(t) - t) + g'(u(t))[g(u(t)) - f(t)] = 0. \tag{2}$$

To see that $\rho(\alpha(t),\beta)/|f(t) - g(t)|$ approaches 1 as t approaches 0, first note that (2) implies that

$$\rho(\alpha(t), \beta) = |f(t) - g(u(t))| [1 + g'(u(t))^2]^{1/2} .$$

Since $u(t)$ approaches 0 as t approaches 0, we may conclude this proof by noting that

$$\begin{aligned} f(t) - g(u(t)) &= [f(t) - g(t)] + g'(v)[t - u(t)] \\ &= [f(t) - g(t)] - g'(v)g'(u(t))[f(t) - g(u(t))] \end{aligned}$$

for some v (depending on t) which lies between t and $u(t)$.

Corollary 5. It readily follows from Lemma 1, that the relation among the components of C-clos V which we have termed equivalence, is a bona fide equivalence relation.

Proof of Proposition 1. We assume in this proof, without loss of generality, that $C - V$ has no components consisting of only one point.

Let l be a continuous linear functional on $A_p(V)$ which vanishes on $M(S)$. We will prove this proposition by showing that such an l must vanish on $A_p(V)$. Pick $\lambda \in L_q(V)$, where $1/p + 1/q = 1$, such that

$$l(f) = \iint_V \lambda(w) f(w)\, du dv \qquad (f \in A_p(V)),$$

and define

$$h(z) = (-1/\pi) \iint_V \lambda(w)[1/(w-z)]\, du dv \qquad (z \in C).$$

We will show that h vanishes on $C - V$, and that there exists a dense subset of $A_p(V)$ consisting of functions g such that

$$\iint_V [|g(z)h(z)|/\rho(z, C-V)]\, dx dy < \infty. \tag{3}$$

It follows from the proof of the theorem in Bell [*1*], as amended in Bell [*2*], that for such a g, $l(g) = 0$, so $l \equiv 0$ on $A_p(V)$.

Assume that the hypothesis of the proposition holds. We now show that h vanishes on $C - V$. It follows from the proof of the theorem in Bell [*1*] that h vanishes on S, on the unbounded component of C-clos V, and on any component of C-clos V which is equivalent to one on which h vanishes. Thus, to show that h vanishes on $C - V$, it is sufficient to show that if Y is a component of C-clos V which satisfies ii or iii, then h vanishes on Y.

Recall that there exist positive constants N and ε such that

$$|h(z) - h(w)| \leqq N|z-w|^{\varepsilon} \qquad (z, w \in C). \tag{4}$$

Let Y be the set of points strictly within a smooth, simple closed curve α. As proven in Warschawski [*6*], one can choose a homeomorphism T of $\{z: |z| \leqq 1\}$ onto clos V which has a complex derivative $T'(z)$ for $|z| \leqq 1$ (the limit of the difference quotient is taken over $\{z: |z| \leqq 1\}$), where $T': \{z: |z| \leqq 1\} \to C$ is continuous. If Y satisfies ii, using (4) and a change of variables one sees that

$$\int_{-\pi}^{\pi} \log |h \circ (e^{it})|\, dt = -\infty. \tag{5}$$

But since $h \circ T$ is continuous on $\{z: |z| \leqq 1\}$, and holomorphic on $\{z: |z| < 1\}$, (5) contradicts the Jensen inequality, unless $h \circ T$ is identically zero on $\{z: |z| \leqq 1\}$.

Next assume instead that Y satisfies iii. Let $z_1, z_2, z_3, \cdots$ be a sequence of distinct points of $S \cap Y$ such that

$$\sum_{j=1}^{\infty} \rho(z_j, \partial Y) = \infty, \tag{6}$$

and let $w_j = T^{-1}(z_j)$ for $j = 1,2,3,\cdots$. Since $h \circ T$ is continuous on $\{z:|z| \leqq 1\}$, and $h \circ T(w_j) = 0$ for $j = 1,2,3,\cdots$ (where each $|w_j| < 1$), as proven in Hoffman [5, p. 63], if $h \circ T$ is not identically zero on $\{z:|z| \leqq 1\}$ then

$$\sum_{j=1}^{\infty} 1 - |w_j| < \infty.$$

We will now show that if (6) holds then

$$\sum_{j=1}^{\infty} 1 - |w_j| = \infty. \tag{7}$$

For any points w and w' in $\{w:|w| \leqq 1\}$,

$$|T(w) - T(w')| = \left| \int_L T'(\zeta)d\zeta \right| \leqq M\left|w - w'\right|,$$

where L is the line segment from w' to w, and

$$M = \sup\{|T'(w)|:|w| \leqq 1\} < \infty.$$

It follows that (6) implies (7), completing our proof that h vanishes on $C - V$.

Comparing the above with the proof of the theorem referred to, we see that we can already deduce our proof for the case $p = 1$. So from now on in this proof, let us assume that $1 < p < 2$.

We now show that $A_p(V)$ has a dense subset consisting of functions f such that for any point ζ of the boundary of V,

$$|g(z)| \leqq \text{const}|z - \zeta|^{-1/p} \quad \text{in some neighborhood of } \zeta. \tag{8}$$

If V is not connected, let the components be $V_1, V_2, V_3, \cdots$, and let $f \in A_p(V)$. f is the limit in $A_p(V)$ of functions $F_1, F_2, F_3, \cdots$, where $F_n(z) = f(z)$ if $z \in V_j$ for $j = 1,2,\cdots,n$; $F_n(z) = 0$ if $z \in V_j$ for $j = n+1, n+2, \cdots$. Hence it is sufficient to demonstrate this step for connected V's. We may further reduce matters to the case in which V is simply connected: Let $H_0, H_1, \cdots, H_n$ denote the components of $K - V$, where H_0 is unbounded, and where K is the Riemann sphere. Using the Cauchy integral representation of f over $n+1$ Jordan curves in V which approximate $\partial H_0, \partial H_1, \cdots, \partial H_n$, we obtain functions $f_0, f_1, \cdots, f_n$ such that each f_j is holomorphic on $H_j^* = K - H_j$, and $f = f_0 + f_1 + \cdots + f_n$. We then approximate each f_j by a function of the required type: Approximate f_0 in $A_p(H_0^*)$. If $j \in \{1,2,\cdots,n\}$, we proceed as follows. First, pick a non degenerate simple arc α in H_j. For simplicity in notation, assume that -1 and 1 are the endpoints of α. Consider the map $T(w) = (w + 1/w)/2$ of a bounded Jordan domain Δ onto the complement of α in the Riemann sphere, where $T(0) = \infty$, and $Y(\partial\Delta) = \alpha$. The problem is then reduced to finding a suitable sequence $\{\phi_k\}_{k=1}^{\infty}$ which converges to

$$f_j \circ T(w)T'(w)^{2/p}w^4$$

in $A_p(U(H_j^*))$, for then

$$\{\phi_k \circ U(z)U'(z)^{2/p}U(z)^{-4}\}_{k=1}^{\infty}$$

converges to f_j in $A_p(V)$, where $U = T^{-1}$.

If $U(H_j^*)$ happens to be a Jordan domain, it is known (see, for example, Farrell [4]) that the polynomials are dense in $U(H_j^*)$, which would then complete this part. The worst that $U(H_j^*)$ could be is of the form $D - \bigcup_{j=1}^{m} \alpha_j$, where D is a Jordan domain, and $\alpha_1, \alpha_2, \cdots, \alpha_m$ are arcs which all meet the boundary of D. For example, let

$$U(H_j^*) = \{z: |z| < 1\} - [1/2, 1].$$

Using a branch $\tau(z)$ of $(z - 1/2)^{1/2}$, we can map $U(H_j^*)$ conformally onto a Jordan domain Δ'. Consider the isometry of $A_p(\Delta')$ onto $A_p(U(H_j^*))$ in which an element ϕ of $A_p(\Delta')$ corresponds to $\phi \circ \tau\tau'^{2/p}$. Again appealing to Farrell [4], we see that

$$\{\phi \circ \tau\tau'^{2/p}: \phi \text{ is a polynomial}\} \tag{9}$$

is a dense subset of $A_p(U(H^*))$. We would then use the method of the previous paragraph to approximate f_j in $A_p(V)$ by functions corresponding to those in (9). Should $U(H_j^*)$ have more cuts than the case discussed, instead of τ we would use a composition of maps like τ, each of which opens up a particular cut.

We may now conclude this proof by noting that (3) follows from (8) together with

$$|h(z)| \leqq N\rho(z, C - V)^{1-2/q} \qquad (z \in C), \tag{10}$$

where N is some constant, and $1/p + 1/q = 1$.

(10) is shown in Bell [1].

Proof of Proposition 2. Let S, V, and X satisfy i, ii, and iii of Proposition 2, and let α denote the smooth, simple closed curve ∂X. Then we can find a smooth, simple closed curve β which encloses α (α and β may meet), which satisfies the following three conditions:

(11) All points of S are either in $\alpha \cup \beta$, or outside β.

(12) The open set D of points lying between α and β, is contained in V.

$$\int_{\alpha} \log \rho(z,\beta) ds > -\infty \tag{13}$$

This proof can be thought of as having two steps (which will be proven simultaneously):

A. β may be chosen so that every point of S lies either on or outside β.

B. There exists a smooth, simple closed curve γ between α and β such that

$$\int_{\gamma} \log \rho(z,\alpha \cup \beta) ds > -\infty .$$

The proof of Proposition 2 may then be deduced from Corollary 2 of Bell [*1*].

Let β_1 satisfy (11), (12), and (13). We will now show how to construct a β which also satisfies A, in a neighborhood of a point z of $a \cap S$. As in the proof of Lemma 1, we may assume that $z = 0$, and that for some ε, $0 < \varepsilon < 1$, $\alpha(t) = t + if(t)$, and $\beta_1(t) = t + ig(t)$ for $|t| < \varepsilon$, where $f(t)$ and $g(t)$ are twice continuously differentiable and $f(t) \leqq g(t)$ for $|t| \leqq \varepsilon$, $f(0) = 0 = g(0)$, and $f'(0) = 0 = g'(0)$. Let

$$F = \{t : \alpha(t) \in S \text{ and } |t| < \varepsilon\}.$$

It is easily proven, choosing ε closer to 0 if necessary, that

$$\int_{-\varepsilon}^{\varepsilon} \log \rho(t,F) dt > -\infty .$$

We next construct a function h which is twice continuously differentiable on $[-\varepsilon,\varepsilon]$, $0 \leqq h(t) \leqq 1$ for $|t| \leqq \varepsilon$, $h(t) = 0$ for $t \in F$, and

$$\int_{-\varepsilon}^{\varepsilon} \log h(t) dt > -\infty .$$

Let $I_1, I_2, I_3, \cdots$ be an enumeration of the components of $[-\varepsilon,\varepsilon] - F$ and let μ_j denote half the length of I_j for $j = 1,2,3,\cdots$. Clearly $\sum_{j=1}^{\infty} \mu_j = \varepsilon$, and

$$-\infty < \int_{-\varepsilon}^{\varepsilon} \log \rho(t,F) dt = 2 \sum_{j=1}^{\infty} \int_{0}^{\mu_j} \log t \, dt .$$

Since

$$\int_0^{\mu_j} \log t\,dt = \mu_j \int_0^1 \log(t\mu_j)\,dt$$

$$= \mu_j \int_0^1 \log t\,dt + \mu_j \log \mu_j,$$

it follows that

$$-\infty < \int_{-\varepsilon}^{\varepsilon} \log \rho(t,F)dt$$

$$= 2\varepsilon \int_0^1 \log t\,dt + 2 \sum_{j=1}^{\infty} \mu_j \log \mu_j .$$

Now consider the function $\phi\colon[0,1] \to [-1,1]$ whose graph consists of the line segments connecting each successive pair of points in the sequence

$$0,\ 1/8 + i,\ 3/8 - i,\ 1/2,\ 5/8 - i,\ 7/8 + i,\ 1,$$

as depicted in Figure 3(a). Given a positive constant μ define the functions

$$\psi_\mu(x) = \int_0^x \mu\phi(t/\mu)dt \qquad (0 \leqq x \leqq \mu),$$

and

$$\omega_\mu(x) = \int_0^x \psi_\mu(t)\,dt \qquad (0 \leqq x \leqq \mu).$$

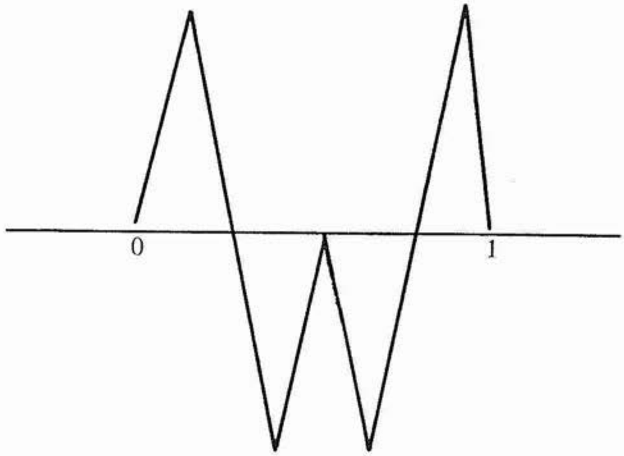
The graphs of ψ_μ and ω_μ are shown in Figures 3(b) and 3(c) respectively.

$$\psi_\mu(x) = \mu \int_0^x \phi(t/\mu)dt = \mu^2 \int_0^{x/\mu} \phi(t)dt = \mu^2\psi_1(x/\mu).$$

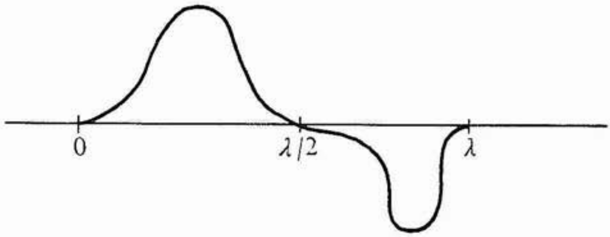
$$\omega_\mu(x) = \mu^2 \int_0^x \psi_1(t/\mu)dt = \mu^3\omega_1(x/\mu).$$

$$\int_0^{2\mu} \log \omega_{2\mu}(t)dt = 6\int_0^{\mu} \log \mu dt + 2\int_0^{\mu} \log \omega_1(t/\mu)dt$$

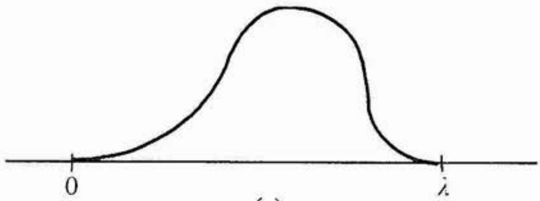
$$= 6\mu \log \mu + 2\mu \int_0^1 \log \omega_1(t)\,dt.$$



(a)



(b)



(c)

Figure 3.

We can now define the function h as follows: If $t \in F$ let $h(t) = 0$. If $t \in I_j = (a_j, b_j)$, let $h(t) = \omega_{2\mu_j}(t - a_j)$. It then follows from the above that h is a twice continuously differentiable function on $[-\varepsilon, \varepsilon]$, $0 \leqq h(t) \leqq 1$ for $|t| \leqq \varepsilon$, $h(t) = 0$ if $t \in F$, and

$$\int_{-\varepsilon}^{\varepsilon} \log h(t) > -\infty .$$

Then define

$$\beta(t) = t + i[f(t) + h(t)(g(t) - f(t))],$$

and

$$\gamma(t) = t + i[f(t) + h(t)(g(t) - f(t))/2],$$

for $|t| \leqq \varepsilon$. As β and γ have the required properties for sufficiently small $\varepsilon > 0$, our proof of Proposition 2 is completed.

Proof of Proposition 3. As in the proof of Proposition 2, we may assume that X is the set of points inside a smooth, simple closed curve α, α lies inside a smooth, simple closed curve β, $\alpha \cap S \subset \beta$,

$$\int_\alpha \log \rho(z, \beta) ds > -\infty,$$

and the open set of points between α and β is contained in V. Let δ and S satisfy all applicable hypotheses of this proposition. The main steps of this proof are:

(A) β may be chosen so that in addition to having the properties stated in the first sentence of this proof, it also satisfies

$$\Sigma\, \rho(z, \beta) < \infty \qquad (z \in S \cap \operatorname{clos} X). \tag{14}$$

(B) There exists a bounded holomorphic function ϕ on the domain D consisting of all points inside β, such that $D \cap S$ is the set of zeros of ϕ.

This proof may then be completed by observing that

$$\{\phi f : f \in M(S) \text{ and } \|f\| < 1\}$$

is a normal family on D. This may be deduced from a slight modification of Proposition 2.

A proof of (A) may be obtained by first localizing our study to a neighborhood of a point $\alpha \cap \beta$, as in Proposition 2. Then do some routine manipulations of the type found in the previous proofs.

To establish (B) we first pick a conformal map T of D onto the unit disk Δ, and consider $z \in X \cap S$. Choose $\zeta \in \partial X$ so that $\rho(z, \partial D) = |\zeta - z|$. Then

$$1 - |T(z)| \leqq |T(\zeta) - T(z)| = \left| \int_L T'(w) dw \right| \leqq c|\zeta - z| = c\rho(z, \partial D), \tag{15}$$

where L is the line segment from z to ζ, and where c is any upper bound of $|T'|$ on D. (That T' is continuous on $\operatorname{clos} D$ is shown in Warschawski [6].) (14) and (15) imply that

$$\Sigma\, 1 - |a| < \infty \qquad (a \in T(S)),$$

so, as proven in Hoffman [*5*, p. 64], one can construct a bounded function holomorphic function Ψ on Δ whose set of zeros is $T(S) \cap \Delta$, through the use of Blaschke products. We may then let $\phi = \Psi \circ T$, completing our proof of Proposition 3.

Added in Proof. Several improvements have since been obtained by the author. For example, if V is sufficiently regular the hypotheses of Proposition 1 are both necessary and sufficient for the conclusion.

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